7

Winding Numbers and Topology

In this chapter we shall investigate a simple but immensely powerful concept—the number of times a loop winds around a point. In Chapter 2 we saw that this concept was needed to understand multifunctions, and in the next chapter we will see that it plays an equally crucial role in understanding complex integration. However, only the first two sections [up to (2)] of the present chapter are actually a prerequisite for that work; the rest may be read at any time. If you are in a rush to learn about integration, you may wish to skip the rest of the chapter and return to it later.

I Winding Number

1 The Definition

As the name suggests, the *winding number* $\nu(L, 0)$ of a closed loop L about the origin 0 is simply the *net* number of revolutions of the direction of z as it traces out L once in its given sense. A nut on a bolt admirably illustrates the concept of "net rotation": spin the nut this way and that way for a while; the final distance of the nut from its starting point measures the net rotation it has undergone.

Figure [1] shows six loops and their corresponding winding numbers. You can verify these values by starting at a random point on each curve and tracing it out with your finger: starting with zero, add one after each positive (= counterclockwise) revolution of the vector connecting the origin to your finger, and subtract one after

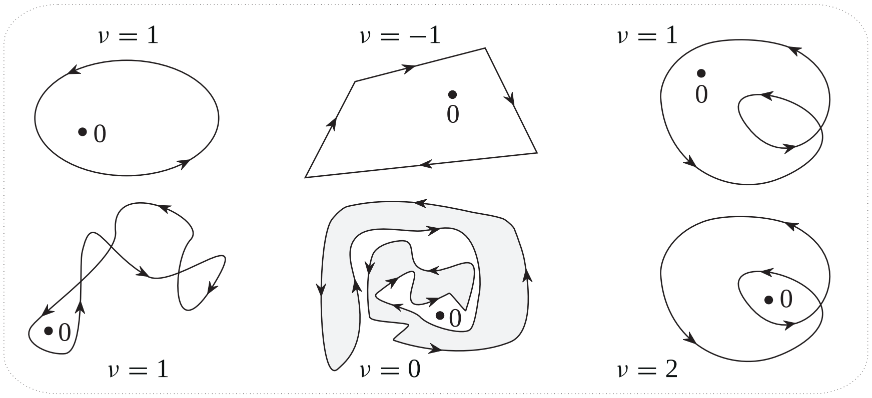

Figure [1]

each negative (= clockwise) revolution. When you have returned to your starting point, the final count is the winding number of the loop.

It is often useful to consider the winding number of a loop about a point p other than the origin, and this is correspondingly written $\nu(L, p)$. Instead of counting the revolutions of z, we now count those of $(z - p)$. For example, the shaded region in [1] can be defined as all the positions of p for which $\nu(L, p) \neq 0$. Try shading this set for the other loops.

2 What does "inside" mean?

A loop is called *simple* if it does not intersect itself; for example, circles, ellipses, and triangles are all simple. Although a simple loop can actually be very complicated [see Ex. 1] it seems clear, though it is hard to prove, that it will divide the plane into just two sets, its inside and its outside. However, in the case of a loop that is not simple, such as [2], it is no longer obvious which points are to be considered

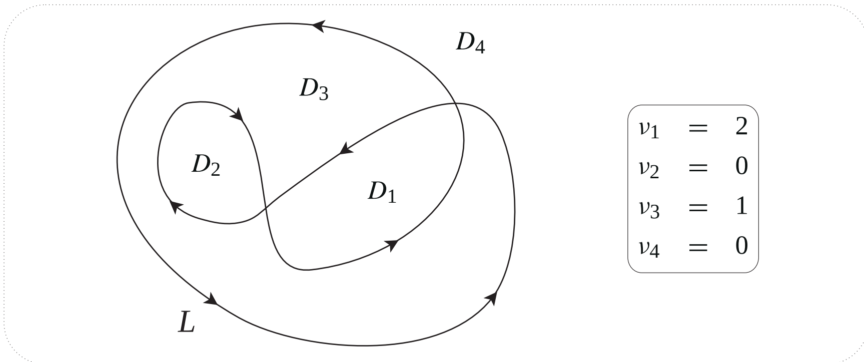

Figure [2]

inside the loop, and which outside. The winding number concept allows us make the desired distinction clearly.

A typical loop such as L will partition the plane into a number of sets D_j (four in this case). If the point p wanders around within one of these sets then it seems plausible that the winding number $\nu(L, p)$ remains constant. Let's check this.

Concentrate on just a short segment of L. As z traverses it, the rotation of $(z - p)$ will depend continuously on p unless[1] p crosses L. In other words, if we move p a tiny bit then the rotation angle will likewise only change a tiny bit. Since the winding number of L is just the sum of the rotations due to all its segments, it follows that it too depends continuously on the location of p: a tiny movement of p to $\widetilde{p}$ can only produce a tiny change $[\nu(L, \widetilde{p}) - \nu(L, p)]$ in the winding number. But since this small difference is an *integer*, it must be exactly 0. Done.

Since L winds round each point of D_j the same number of times, it follows that we can attach a winding number ν_j to the set as a whole. Verify the values of ν_j given in the figure.

[1] Consider the behaviour of the rotation due to a short segment of L as p crosses it.

The "inside" can now be *defined* to consist of those D_j for which $\nu_j \neq 0$, while the remaining D_j constitute the "outside". Thus in [2] we find that $D_1 \cup D_3$ is the inside, while $D_2 \cup D_4$ is the outside.

The "correctness" of this definition will become apparent in the next chapter.

3 Finding Winding Numbers Quickly

In [2] we found the winding numbers directly from the definition: we strenuously followed the curve with our finger (or eye) and counted revolutions. For a really complicated loop this could literally become a headache. We now derive a much quicker and more elegant method of visually computing winding numbers.

If a point r moves around without crossing a loop K then $\nu(K, r)$ remains constant, but what happens when the point *does* cross K? Consider [3]. On the far left, close to the loop K, is the point r; the rest of K is off the picture, and the number of times it winds round r is $\nu(K, r)$. The time-lapse pictures in [3] show r moving towards the loop, which itself deforms so as to avoid being crossed, finally ending up at the point s.

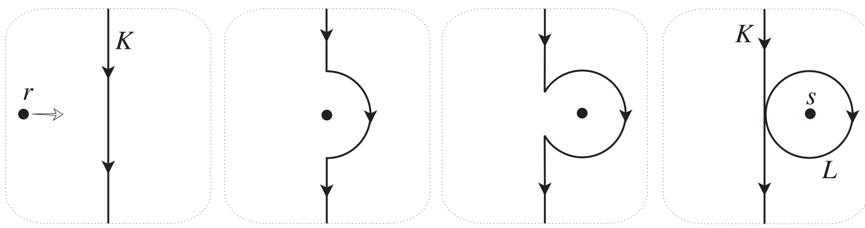

Figure [3]

Now since the moving point never crosses the loop, the winding number remains constant throughout the process. But on the far right, the new loop can be thought of as the union of the old loop K, together with the new circle L. Thus,

$$\begin{aligned} \nu(K, r) &= \nu(K, s) + \nu(L, s) = \nu(K, s) - 1 \\ \Rightarrow \nu(K, s) &= \nu(K, r) + 1\,. \end{aligned}$$

Imagining ourselves at r, looking towards K as we approach it, we may express this result in the form of the following very useful *crossing rule*:

If K is moving from our left to our right [our right to our left] *as we cross it, its winding number around us increases* [decreases] *by one.* (1)

Using this result, it is incredibly quick and easy to find the ν_j's for even the most complicated loop. Try it out on [2]. Starting your journey well outside L, where you know that the winding number is zero, move from region to region, using crossing rule (1) to add or subtract one at each crossing of L.

An immediate consequence of this idea is a connection between $n = \nu(K, p)$ and the number of intersection points of K with a ray emanating from p. Suppose that the ray is in general position in the sense that it doesn't pass through any self-intersection points of K, nor is it tangent to K. If a point q on this ray is sufficiently

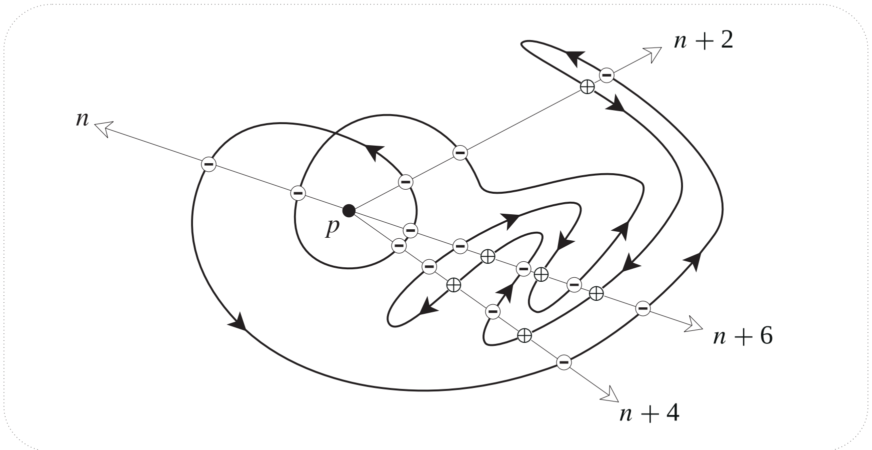

Figure [4]

distant from p then clearly K cannot wind around it; thus as we move along the ray from p to q the winding number changes by n. But the winding number only changes when we cross K, and only one unit per crossing. The ray must therefore intersect K at least $|n|$ times. However, in addition to these $|n|$ necessary crossings there may be additional cancelling *pairs* of crossings. In general, then, the number of intersection points will be $|n|$, or $|n|+2$, or $|n|+4$, etc. Figure [4] illustrates these possibilities for a case in which $n = 2$, each intersection point being marked with $\oplus$ or $\ominus$ according as the winding number increases or decreases as it is crossed.

II Hopf's Degree Theorem

1 The Result

We have discussed the fact that for a fixed loop and a continuously moving point, the winding number only changes when the point crosses the loop. But it is clear that the same must be true of a fixed point and a continuously moving loop: the winding number of the evolving loop can only change if it crosses the point, and it changes by ± 1 according to the same crossing rule as before. Thus if a loop K can be continuously deformed into another loop L without ever crossing a point p, the winding numbers of K and L around p will be equal.

It is natural to ask if the converse is also true: if K and L wind round p the same number of times, is it always possible to deform K into L without ever crossing p? This is certainly a more subtle question, but by drawing examples you will be led to suspect that it is true. In this section we will confirm this hunch, so establishing that

> *A loop K may be continuously deformed into another loop L, without ever crossing the point p, if and only if K and L have the same winding number round p.* (2)

At the end of the next chapter, this will turn out to be the key to understanding one of the central results of complex analysis.