

Now let the squares of the grid shrink so as to completely fill the interior R of C . Using (3) and (4) and replacing the sum over squares by a double integral over infinitesimal areas dA , we obtain *Gauss' Theorem*,

$$F[\mathbf{X}, C] = \iint_R [\nabla \cdot \mathbf{X}] dA, \quad (7)$$

and *Stokes' Theorem*,

$$W[\mathbf{X}, C] = \iint_R [\nabla \times \mathbf{X}] dA. \quad (8)$$

From these we see that if the divergence and curl vanish everywhere in R then the flux and work for C also vanish, as was required.

Again following the logic in Chapter 8, consider what happens to the flux and work as we continuously deform a closed contour, or an open contour with fixed end points. You should be able to see that (7) and (8) imply two deformation theorems:

If the contour sweeps only through points at which the divergence vanishes, the flux does not change. (9)

If the contour sweeps only through points at which the curl vanishes, the work does not change. (10)

II Complex Integration in Terms of Vector Fields

1 The Pólya Vector Field

Consider

$$\int_K H(z) dz$$

from the vector field point of view. See [7]. In forming a Riemann sum with terms $H dz$ we now have the minor advantage that $H = |H| e^{i\beta}$ and $dz = e^{i\alpha} ds$ are not drawn in separate planes, as they were in Chapter 8. However, we still face the problem that $H dz = |H| e^{i(\alpha+\beta)} ds$ involves the *addition* of angles, which is not easy to visualize. Just as it is more natural to subtract vectors [yielding connecting vectors] than to add them, so it is also more natural to subtract angles, for this yields the angle contained between two directions.

The simple and elegant solution to our problem is to consider a new vector field: instead of drawing $H(z)$ at z we draw its *conjugate* $\overline{H(z)} = |H| e^{-i\beta}$. We shall call this the *Pólya vector field* of H . Before showing how this solves our problem, let us offer (i) a caution and (ii) a reassurance:

- (i) The Pólya vector field of H is *not* obtained by reflecting the picture of the ordinary vector field for H in the real axis, for this would attach $\overline{H(z)}$ to \bar{z} instead of z . This will become very clear if you (or your computer) draw the

Pólya vector fields of z and z^2 , for example. Comparison with [1], p. 450, reveals that the resulting phase portraits (not the vector fields themselves) are identical to those of $(1/z)$ and $(1/z^2)$. This is because \bar{z}^n points in the same direction as $(1/z^n)$.

- (ii) As we will see in a moment, much is gained by representing H by its Pólya vector field, but we also wish to stress that nothing is lost: the new field contains exactly the same information as the old one. For example, it is clear that the index of a loop L merely changes sign when we switch to the Pólya vector field:

$$\mathcal{I}_{\bar{H}} [L] = -\mathcal{I}_H [L].$$

Thus an n^{th} order root of an analytic H still shows up clearly in its Pólya vector field as a singular point, but now with index $-n$ instead of n . Likewise, a pole of order m produces a singular point of index m instead of $-m$.

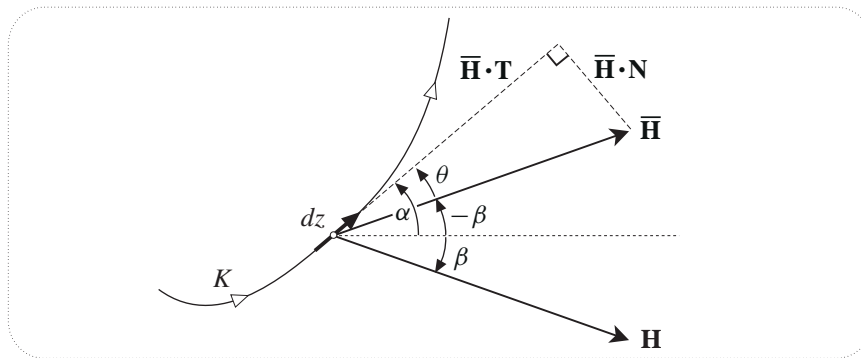


Figure [7]

Returning to integration, the great advantage of the Pólya vector field is that the angle θ that it makes with the contour (see [7]) is given by $\theta = \alpha - (-\beta)$, and this is precisely the angle we were trying to visualize—the angle of the term $H dz$ in the Riemann sum. Better still, we find that

$$\begin{aligned} H dz &= |\bar{H}| e^{i\theta} ds \\ &= [|\bar{H}| \cos \theta + i |\bar{H}| \sin \theta] ds \\ &= [\bar{H} \cdot \mathbf{T} + i \bar{H} \cdot \mathbf{N}] ds. \end{aligned}$$

Thus the real and imaginary parts of each term in the Riemann sum are the work and flux of the Pólya vector field for the corresponding element of the contour. We have thus discovered a vivid interpretation (due to Pólya²) of the complex integral of H in terms of the work and flux of its Pólya vector field along the contour:

²See Pólya and Latta [1974].

$$\int_K H(z) dz = W[\bar{\mathbf{H}}, K] + i F[\bar{\mathbf{H}}, K]. \quad (11)$$

This interpretation is rendered particularly useful by the fact that a computer can instantly draw the Pólya vector field of any function you wish to integrate. You can then quickly get a feel for the value of the integral by looking at how much the field flows along and across the contour. For example, the integral of $(\bar{z}^2 z)$ along the line-segment from $1 - i$ to $1 + i$ is clearly a positive multiple of i . Why? For more on the nitty-gritty of estimating integrals with (11), see Braden [1987].

Our interest in (11) will be less in this practical aspect, and more in its theoretical import: ideas about flows and force fields can shed light on complex integration, and vice versa. In what follows we shall give examples in both directions.

2 Cauchy's Theorem

Given a picture of the vector field of a complex mapping $H(z) = u + iv$, how can we tell whether or not H is *analytic*? To my knowledge there is no satisfactory answer to this question as posed. However, there is an answer if we instead look at the Pólya vector field, and it is an answer that exhibits a beautiful connection between physics and complex analysis:

$$\textit{The Pólya vector field of } H \textit{ is divergence-free and curl-free if and only if } H \textit{ is analytic.} \quad (12)$$

The verification is a simple calculation:

$$\nabla \cdot \bar{\mathbf{H}} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} u \\ -v \end{pmatrix} = \partial_x u - \partial_y v,$$

and

$$\nabla \times \bar{\mathbf{H}} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \times \begin{pmatrix} u \\ -v \end{pmatrix} = -(\partial_x v + \partial_y u).$$

Thus the divergence and curl of $\bar{\mathbf{H}}$ will both vanish if and only if the Cauchy-Riemann equations are satisfied. Note for future use that these two equations are really two aspects of a single complex equation,

$$i \partial_x H - \partial_y H = \nabla \times \bar{\mathbf{H}} + i \nabla \cdot \bar{\mathbf{H}}, \quad (13)$$

the vanishing of the LHS being the compact form of the CR equations.

With this connection established, we now have a second, *physical* explanation of Cauchy's Theorem which is scarcely less intuitive than the geometric one in Chapter 8. For if H is analytic everywhere inside a simple loop K bounding a region R , its Pólya vector field in R will have (as a flow) zero flux density and (as a force field) zero work density. This means that there is no net flux of fluid out of R , and that a puck fired round K returns with its kinetic energy unchanged. From (11) we see that the integral of H round K must vanish.

A more mathematical version of this physical explanation was given at the end of the last section in terms of the theorems of Gauss and Stokes. Restating that argument in the present context, for a simple loop K bounding a region R , substitution of (7) and (8) into (11) yields

$$\oint_K H(z) dz = \iint_R [\nabla \times \bar{\mathbf{H}}] dA + i \iint_R [\nabla \cdot \bar{\mathbf{H}}] dA, \quad (14)$$

which vanishes if $\bar{\mathbf{H}}$ is curl-free and divergence-free in R .

3 Example: Area as Flux

As a fun and instructive example let us reconsider the result

$$\oint_K \bar{z} dz = 2iA \quad (15)$$

in the light of the physically intuitive theorems of Gauss and Stokes.

Observe that the Pólya vector field of $H(z) = \bar{z}$ is $\bar{H}(z) = z$, which flows radially outwards from the origin, like a source. However, unlike a source, here the speed of the flow *increases* with distance, making it clear that this flow cannot be divergence-free. Indeed, calculating its flux density, we find that

$$\nabla \cdot \bar{\mathbf{H}} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2.$$

In other words, in each unit of time, 2 units of fluid are being pumped into each unit of area. The flux of fluid out of K is therefore $2A$. On the other hand the flow *is* curl-free:

$$\nabla \times \bar{\mathbf{H}} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = 0,$$

so there is no circulation round K . Inserting these facts into (11) we obtain (15).

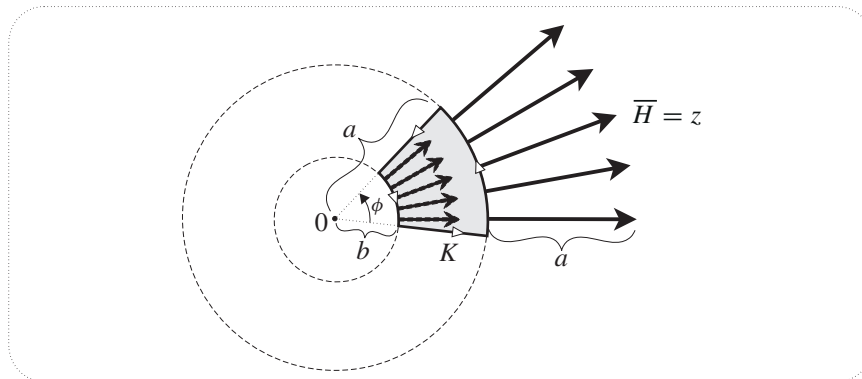


Figure [8]

Figure [8] is a concrete example of this new way of looking at (15), the shape of K having been chosen so as to make the values of the circulation and flux obvious.

Clearly $\overline{H(z)} = z$ has no circulation along either of the arcs, and it has equal and opposite circulations along the line-segments. The total circulation round K therefore vanishes. Equally clearly, there is no flux across the line-segments, but there is across the arcs. The larger arc has length $a\phi$ and the speed of the fluid crossing it is a , so the flux across it is $a^2\phi$; similarly, for the smaller arc it is $b^2\phi$. Thus,

$$F[z, K] = (\text{fluid out}) - (\text{fluid in}) = 2 \left[\frac{1}{2} a^2 \phi - \frac{1}{2} b^2 \phi \right] = 2 \text{ (shaded area)}.$$

Before moving on, let us clear up a paradoxical feature of the vector field z : fluid is being pumped in *uniformly* throughout the plane, and yet the flow appears to radiate from one special place, namely, the origin. The resolution (see [9]) lies in the trivial identity $z = z_0 + (z - z_0)$, which says that the flow from the origin is the superposition of the sourceless, irrotational field z_0 and a copy of the original flow, but now centred on the arbitrary point z_0 instead of the origin.

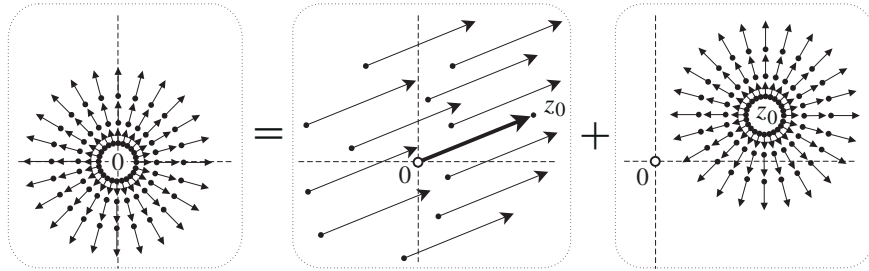


Figure [9]

4 Example: Winding Number as Flux

Next, let us see how the Pólya vector field also breathes fresh meaning into the fundamentally important formula

$$\oint_L \frac{1}{z} dz = 2\pi i \nu [L, 0]. \tag{16}$$

According to (11),

$$\oint_L \frac{1}{z} dz = W[(1/\bar{z}), L] + i F[(1/\bar{z}), L].$$

But the Pólya vector field $(1/\bar{z})$ is an old friend—it is a source of strength 2π located at the origin.

Figure [10] illustrates the intuitive nature of the result from the new point of view. If a loop does not enclose the source, just as much fluid flows out as in; if a simple loop does enclose the source, it intercepts the full 2π of fluid being pumped