Spillovers and Growth in a Local Interaction Model*

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Abstract

This theoretical model investigates the extent of the effects of local technological spillovers on growth. We add to a growth model with vertical quality ladder innovations the assumption that firms are exogenously located around a circle and technological spillovers affecting R&D efficacy occur between neighbors. As a result, the presence of local spillovers makes it optimal for a firm to innovate intensively only if the neighbors do so. However, the effects of the spillovers are ambiguous. They increase the probability of success of innovation, but also the obsolescence rate and, under specified conditions, they decrease the benefits of a sector's current research. Moreover, spillovers do not produce clear effects on either growth or welfare. On one hand positive spillovers improve the quality of local goods—hence growth—but, on the other, a high level of positive spillovers generate discrepancies across intermediate sectors which reduce the level of household welfare.

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1 Introduction

Economic activities are not evenly distributed across space: some regions display a high concentration of activities—agglomeration—and benefit from the resulting wealth, others struggle. We can trace this phenomenon within a country[1] as well as across countries.[2]

One powerful explanation relates agglomeration to the naturally uneven distribution of natural resources. Firms (and more in general, economic activities) have a tendency to concentrate in regions exogenously endowed with natural resources or in proximity to natural ways of communication. Presence of natural resources seems the major force driving the location of activities especially in the past, e.g. during the Industrial Revolution, at the time of the Roman empire, and the reason behind Florida attracting many retired people (Ottaviano-Thisse, 2003).

This explanation is not satisfying when we try to understand the rise of concentration of activities less dependent on natural advantages. Striking cases include Silicon Valley (Saxenian, 1994) and the Italian industrial districts; but also numerous other less-known situations as the Wales industrial reconversion planned through the creation of local (more or less spontaneous) agglomerations of firms (Cooke and Morgan, 1998; OECD, 1996). Without denying the importance of exogenous endowments, recent economic studies have focused on agglomeration as the result of human activities. The idea is that, even within a fairly balanced regional system, initial asymmetries can get magnified by cumulative causation and produce a *core and periphery* structure, i.e. an unequal distribution of activities that leads most of them to concentrate in a particular bounded area (the core) and leave the other part empty (the periphery). In these models, the *cumulative causation* between the location of firms and workers yields self-reinforcing movements explaining local specialization and affecting regional growth and decline. In this framework, increasing returns (often in the form of firms serving larger and larger markets) play an important role, while the scarcity of local raw materials does not matter since production factors may be imported from elsewhere. Hence, the lack of natural resources is less crucial for economic development (Krugman, 1991; Krugman *et al.*, 1999).

Agglomeration has been addressed by Fujita and Thisse (2002) as the territorial counterpart of economic growth. In order to study its dynamics they propose a model of endogenous growth for an economy with two regions. Their model relies on the combination of the building blocks of Krugman's core-periphery model and Grossman-Helpman-Romer's model of endogenous growth with horizontal differentiation. In addition, they introduce an R&D sector that uses skilled labor to produce new varieties for the modern sector. Focusing

[1] Within most European countries, e.g. Belgium, Italy or Spain, northern regions tend to be richer than southern regions.

[2] In the European Union, per-capita income is on average higher in the northern countries than in the southern ones. E.g. in 2001 the EU average GDP per capita was 23,230 euro, ranging from 26,640 euro in UK and 33,200 euro in Danmark to about 11,900 euro in Greece and Portugal (source Eurostat).

on the steady state equilibrium, they show that the growth of the economy is related to the spatial organization of the innovation sector across regions. In that sense, the R&D sector appears to be a strong centripetal force since it amplifies the circular causation of the core-periphery structure. In addition, they find that the growth effect can be strong enough to make the level of welfare increase even in the peripheral areas. As in Grossman and Helpman (1991a and 1991b), growth is still driven by the increase in the number of varieties.

The same result, growth driven by variety increases, holds even when we concentrate on a particular case of local agglomeration: the Marshallian districts. As argued in Basevi and Ottaviano (2002), in a district, growth is related to the increase in the varieties of goods produced by the firms. This approach focuses on growth for particular bounded spaces. Proximity matters and enhances the positive effects of investing in R&D (see also Rosenthal and Strange, 2003). This last issue is not new. Usually, the idea of proximity entails the existence of local spillovers and, looking at the microfoundations of local agglomeration (e.g. Saxenian 1994), spillovers are a centripetal (agglomerating) force.

This paper provides a formalization of the aforementioned spillover effects in a growth model. The main goal is to assess to what extent local spillovers can foster economic growth. Our framework develops from a growth model with vertical quality ladder innovations as in Grossman and Helpman (1991a; 1991b), Aghion and Howitt (1992) and Barro and Sala-i-Martin (1995). As in Lucas (1988), we assume that knowledge produces a positive externality, and, as in Romer (1990), that technological change is due to intentional investment decisions. In addition, we impose partial excludability of the technological externality: spillovers do not effect the entire economy but only those locations close to the source of technological advancement—i.e. the most immediate neighbors.

In particular, in our model clusters of innovating firms are due to local externalities that make it optimal for a firm to innovate intensely only if the neighbors do so. A final product is obtained by using different intermediate goods. For each type of a fixed set of intermediate goods, technological innovations take the form of improvements occurring on a quality ladder. Technology is modeled as a locally non-excludable good, so that the probability with which a firm gets a successful innovation is proportional to one's effort and either to the amount of research of the most immediate neighbors or to their relative innovation successes. Research intensity is on average constant across locations where innovation and growth arrive randomly. Economic forces operate as to keep growth similar on average across locations.

One important conclusion is that spillovers may have ambiguous effects. Keeping constant the level of research in a particular sector, a high level of research in one's neighborhood increases the probability of successful innovation, but also the obsolescence rate. In addition, neighbors' future research decreases the benefits of a sector's current research. Moreover, spillovers do not produce clear effects on either growth or welfare. On one hand positive spillovers improve the quality of local goods—hence growth—but, on the other, a high level of positive spillovers generate discrepancies across intermediate sectors which reduce the level of household welfare.

The rest of the paper is organized as follows. Section 2 provides some empirical evidence. Section 3 describes the building blocks of the model. Section 4 presents an analysis of the relationship between spillovers and growth and, finally, Section 5 concludes.

2 Empirical Evidence

The idea that R&D proximity favors R&D intensity has been well documented in the empirical literature (e.g. see Audretsch and Feldman, 1996; Feldman and Audretsch, 1999). To quantitatively disentangle the relationship between technological knowledge and spillover effects, Peri (2003; 2004) takes into account the geographical dimension of the knowledge flows. By using 1975-1996 regional data on patents and their citations in the US and other OECD countries (as a measure of R&D effort), he finds that technological regional leaders may act as a learning source for other regions. Moreover, knowledge generated by these technological leaders is also the most relevant for other regions because of its better quality. It is indeed because of that quality that leaders' knowledge travels across space. For instance, Peri quantifies that if the stock of knowledge were to double in California (referring to 1996 data), innovations in California would rise in the short run by 30%. At the same time, R&D and innovations in Arizona would increase by 30% and 12% respectively, while in the German region of Berlin accessible external knowledge and innovations would increase by 12% and 5.2 % respectively.

With respect to the European Union, the empirical study of Nicolini (2002) on Belgian data finds a tendency, at the district level, for firms investing in R&D to cluster. Figure 1 and 2 depict the distribution of Belgian firms investing in R&D in 1997 for two sectors: 'manufacture of machines and equipment tools (#29)' and 'commerce of means of transport (#50)'. It is immediately apparent that in these two sectors the distribution of firms investing in R&D is far from uniform. Economic activity seems to take place around poles of concentration (or agglomeration), even if the size of such concentrations may vary across sectors and across space.[3] The proposed explanation for this propensity to cluster is that firms, by grouping together, enjoy positive spillovers from the other surrounding firms. These positive effects entail a reduction of the effort that each firm devotes to R&D activity as well as a reduction of the correspondent R&D costs each one has to sustain. In addition to these pecuniary advantages, these positive spillovers generated by the interactions of neighboring firms (either in the same sector or in other sectors whose activities are still strictly connected) increase the likelihood that R&D activity would lead to successful technological innovations.

[3]Figure 1 and 2 represent poles of agglomeration by shadowed areas proportional to their relative firm concentration.

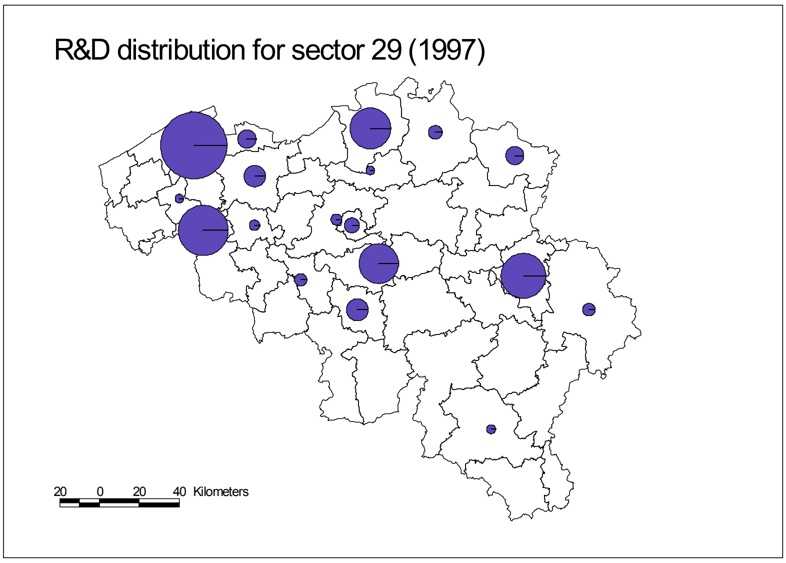

Figure 1: Distribution of Belgian firms investing in R&D. Sector 29: Manufacture of machines and equipment tools. (Source: Nicolini, 2002).

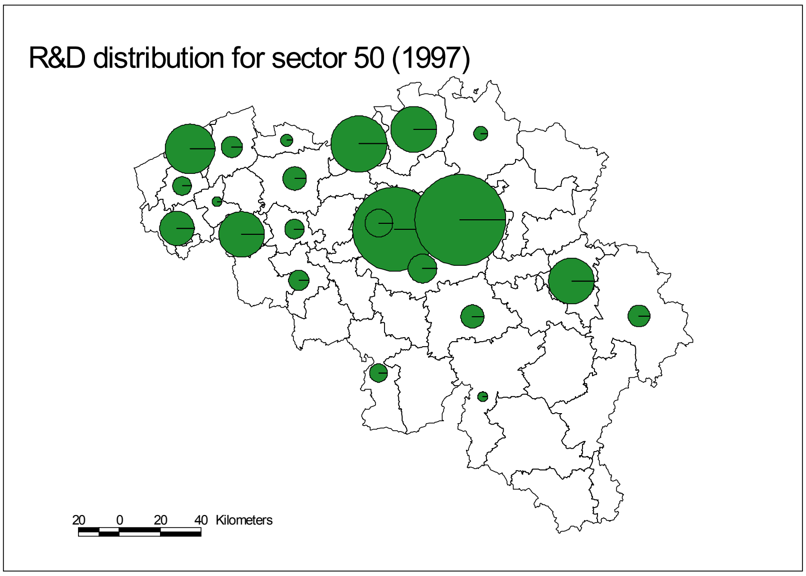

Figure 2: Distribution of Belgian firms investing in R&D. Sector 50: Commerce of means of transport (Source: Nicolini, 2002)

Another example comes from the Italian digital industrial districts.[4] To get a rough estimate of the dimension of the positive effects of these spillovers, we report in Table 1 a comparison of the propensity to innovate and of the relative financial effort between firms belonging and not belonging to a district. Even if the lack of more detailed data prevented us from calculating the significance of these differences, the data provide an initial support to the idea that collaboration among firms generate positive externalities: firms located in a district display a higher propensity to innovate and a lower financial effort than firms located elsewhere.

Table 1: Local firms and innovation (Source: FEDERCOMIN-RUR/CENSIS, 2001)

	Industrial Districts	Local Areas [5]
Firms' propensity to innovate	51%	48.6%
Share of profits invested in innovation projects in 1999	8.6%	9.4%

In conclusion, the empirical evidence reviewed so far supports the idea that knowledge (intended as R&D investment) spreads across space, allowing for the existence of positive externalities. In particular, these spillover effects do not seem to spread evenly across space: proximity matters! This last aspect provides the main motivation for our model.

3 The Model

To study the effect of local technological spillovers on economic growth, we introduce the existence of local externalities between neighboring firms exogenously located on a circle (in the spirit of Salop, 1979), in a framework similar to Barro and Sala-i-Martin (1995), Grossman and Helpman (1991), Aghion and Howitt (1992). As in these models, the economy is comprised of consumers who exchange labor for final goods, producers of final goods and producers of intermediate inputs.

3.1 Final Goods

In the economy there are N final consumption goods. Each final good is produced under perfect competition, using m different intermediate inputs. These intermediate inputs are available in several varieties and continuing improvements and refinements permanently increase their quality. In each intermediate sector j the potential grades are arrayed along a quality ladder with rungs spaced at proportional intervals. Let's assume that the innovation

[4] An industrial district is a concentration of firms belonging to the same sector of activity or to other vertically linked sectors. Usually firms located in an industrial district collaborate significantly with each others and often all of them participate in the development of common projects.

[5] By "local areas" we mean small and rising local forms of agglomeration that cannot be considered industrial districts yet.

of a new variety replaces the old one by raising the quality q_{t_j} by a constant $q > 1$, so that in sector j at the t_j^{th} innovation:

$$q_{t_j+1} = qq_{t_j} \tag{1}$$

with q_{0j} denoting the initial quality in sector j (taken as given). To simplify, we normalize q_{0j} so that each input j begins with $q_{0j} = 1$. Subsequent improvements occur sequentially, jumping discretely one rung at a time, at the levels q, q_1, q_2, and so on. If in j sector t_j improvements have already occurred, then the available grades are 1, q, q_1, q_2, ... , q_{t_j}.

As in Barro and Sala-i-Martin (1995), we assume that each firm i producing a final good uses a Cobb-Douglas production function, i.e. separable in terms of its intermediate inputs:

$$y_i = A(L_i)^{1-\alpha} \sum_{j=1}^{m} (\widetilde{X}_{ij})^{\alpha} \text{ with } \alpha \leq 1,$$

where y_i represents the amount of final good produced by firm i, L_i the amount of unspecialized labor employed by firm i, $\widetilde{X}_{ij} = \sum_{t=0}^{t_j} (q_t x_{i,j,t})$ is an index of the available intermediate inputs from sector j weighted by a function of their quality q_t, and $x_{i,j,t}$ is the amount of intermediate good j of quality q_{t_j} used by firm i. Therefore, the overall input in firm i from sector j is a quality weighted sum of the amount used of each grade. This additivity assumption implies that quality grades within a sector are perfect substitute as inputs of production. Since the latest innovation in each intermediate sector has an efficiency advantage over the prior innovations and a disadvantage relative to the future ones, only the quality of each input that is currently the best will be used, so that the production function simplifies to:

$$y_i = A(L_i)^{1-\alpha} \sum_{j=1}^{m} (q_{t_j} x_{i,j})^{\alpha}. \tag{2}$$

It is important to note that (Eq. 2) displays constant return to scale when quality is kept constant, while increasing return to scale each time the quality steps up one rung.

3.2 Intermediate Goods

We assume that the m intermediate good producers are located around a circle. This spatial assumption allows us to introduce in the model the idea that spillovers occur between neighboring sectors, i.e. they stem only from the activities generated by the sectors located nearby. In this model we are concentrating on nearest inter-sectorial spillovers, but in such a way we are able to model that proximity matters in enjoying positive externalities.

Each intermediate sector is composed of a large number of firms who compete to become the innovation leader. While several firms can be actively pursuing research in each period, only one—the most recent innovator—is allowed to produce and to enjoy a monopoly position. Given the situation, the monopoly power of the most innovative firm in each sector cannot last forever. Even if the other firms in the same sector are not producing in a period, they can still invest in research and become producers in the future. As a result, these competitors are a menace for the leader firm and prevent her from stopping to invest in research.

Quality improvements in the intermediate sectors depend on the successful application of research efforts (namely labor) in a non-deterministic way.[6] Successful innovations arrive at random times.

For each sector, the interval $(t_j + 1) - t_j$ denotes the interval starting with the t_j^{th} innovation and ending just before the $(t_j + 1)^{th}$. During this interval, which has a different length for each values of t_j and for each intermediate sector, the best available quality is q_{t_j}. Even if there are potentially many firms in the same intermediate sector j, only the one that successfully created the j^{th} intermediate good at quality level q_{t_j} retains the monopoly rights to produce it.

Each intermediate (nondurable) good is assumed to be produced with specialized and unspecialized labor under constant return to scale, keeping quality constant. Denote $\overline{L}$ as the total number of workers (and also the total population) shared between the final and the intermediate sectors. Each final sector hires individuals supplying labor just for manufacturing the final good, while each intermediate sector hires labor to work either for manufacturing the intermediate product j or in research. In each period and for each intermediate sector, it holds that

$$l_j = l_j^x + l_j^{R\&D}.$$

Definition 1 *The technology of the production of each intermediate good takes a linear form.*

Hence, given that x_j represents both the number of unit produced and the amount of unskilled labor allocated to manufacturing in sector j, we simplify the previous expression as follows

$$l_j = x_j + n_j, \tag{3}$$

where n_j denotes the amount of skilled labor allocated to R&D in sector j.[7]

[6] As specified in the next section, the utility is linear in consumption and, as in Aghion and Howitt (1992), there is no need to introduce a capital market for risk sharing.

[7] For sake of semplicity, we do not account for a diversification in the qualification of the labor force (between manufacturing and reseach) that usually leads to a differentation of wages.

3.3 Research and Development

Success in R&D is random, so technological progress occurs unevenly in the same sector and across sectors. Consider the number of workers hired in R&D, n_{j,t_j}, as the flow of resources used in R&D by all the potential innovators in sector j when the highest quality ladder reached in each sector j is t_j.[8]

Definition 2 *Innovations in sector j arrive randomly at the following arrival rate*

$$\delta_{j,t_j} = \lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}})n_{j,t_j}, \textit{with } q_{t_{j-1}}, q_{t_j}, q_{t_{j+1}} > 1. \tag{4}$$

The probability of success in research is proportional to the amount of research in sector j, but also to the local characteristics: the amount of research efforts in the two most immediate neighbor locations, and a factor representing their technological progress relative to the progress in sector j.

Define function $\lambda(\cdot)$—representing the productivity of the research technology in sector j—as an increasing function of both the amount of research conducted by the two most immediate neighboring sectors $j-1$ and $j+1$, and their relative technological progress. Hence,

$$\frac{\partial\lambda(\cdot)}{\partial n_{j-1,t_{j-1}}} > 0, \qquad \frac{\partial\lambda(\cdot)}{\partial n_{j+1,t_{j+1}}} > 0, \qquad \frac{\partial\lambda(\cdot)}{\partial \frac{q_{t_{j-1}}}{q_{t_j}}} > 0, \qquad \frac{\partial\lambda(\cdot)}{\partial \frac{q_{t_{j+1}}}{q_{t_j}}} > 0.$$

This assumption embeds the idea of *local positive spillovers* that innovation exerts on neighbors. Being close to technologically advanced firms investing in R&D makes every unit spent in R&D more productive.

Because of the increased productivity of the resources used in research, we cannot determine *a priori* whether this positive externality generates an increase or a decrease in sector j's R&D. Empirical literature seems to confirm the tendency of innovating firms to cluster in order to exploit such positive externalities and, indeed, the strength of local spillovers principally relies on physical proximity (see Audretsch and Feldman, 1996; and Feldman and Audretsch, 1999). In our model, however, the ambiguity of the relation between R&D and the externality stems from the fact, not found in the cited empirical works, that the firm that succeeds in innovating monopolizes the intermediate sector, making previous period innovation obsolete.

[8] Indirectly, this hypothesis refers to another one which we will define later: the level of wages is normalized to one.

3.4 Monopoly Price, Quantity and Profits

Given the hypothesis of free entry into R&D, each potential intermediate producer in sector j has to find the optimal amount of labor resources to allocate to research. Each sector j is in equilibrium when the cost of making research equals the expected benefits. The cost of the research, per unit of time, is the flow of resources invested in R&D by the potential innovator. When the highest quality-ladder achieved in that sector is t_j, this cost is given by $w_{j,t_j} n_{j,t_j}$.

Definition 3 *Given the flow of innovations, in each sector only the highest quality good is produced and its producer enjoys monopoly power.*

The benefit of research occurs only when success arrives. This benefit is given by a stream of monopoly profits to the producer of the highest quality. The duration of the monopoly power is uncertain; it lasts until a competitor breaks through the next improvement. Not all research is successful, therefore the expected returns arrive with probability δ_{j,t_j} (Eq. Eq. 4).

To solve this, we assume that the potential innovator cares only about the expected present value of the stream of profits, and not about the randomness of the returns of her research (as in Barro- Sala-i-Martin, 1995). We begin by computing price, quantity and profit of the monopolist in the intermediate sector j. The producer of the intermediate good x_{j,t_j} using the t_j innovation (either discovered in her own lab or after purchasing the patent from another one) maximizes her profits from the sales of this intermediate good to all the final producers i. For sake of simplicity, we normalize at 1 the level of wages.

Since the final good y_i is produced under perfect competition, the price of the intermediate good that enters as input into the production function of y_i has to equalize its marginal product, which in our Cobb Douglas production function becomes

$$p_{i,j,t_j}(x_{i,j,t_j}) = \frac{\partial y_i}{\partial x_{i,j,t_j}} = AL_i^{1-\alpha} \alpha q_{t_j}^{\alpha} x_{i,j,t_j}^{\alpha-1}, \tag{5}$$

Knowing that only the highest quality of each good is purchased, by aggregating the profit maximizing conditions over all final goods we get the demand function for good x_{j,t_j}

$$x_{j,t_j} = L \left[\frac{A\alpha q^{t_j}}{p_{j,t_j}} \right]^{\frac{1}{1-\alpha}}. \tag{6}$$

Replacing (Eq. 5) and (Eq. 6) into the profits of the leading producer in j, the interior solution, $x_{j,t_j} > 0$, for the first order condition (second order always satisfied) yields the monopoly price

$$p_{j,t} = \frac{1}{\alpha}. \tag{7}$$

Plugging (Eq. 5) into (Eq. 6), the aggregate quantity produced of the leading quality j^{th} intermediate good becomes

$$x_{j,t_j} = L\left[A\alpha^2 q_{t_j}^{\alpha}\right]^{\frac{1}{1-\alpha}}, \tag{8}$$

implying that the evolution of t_j over time in each sector and its divergence across sectors lead to variations of the quality over time and across sectors. Combining (Eq. 5) with (Eq. 6) and taking into account the costs of production, the profits for a temporal monopolist turns out to be

$$\pi_{j,t_j} = \left(\frac{1-\alpha}{\alpha}\right) x_{j,t_j} = \left(\frac{1-\alpha}{\alpha}\right) L\left[A\alpha^2 q_{t_j}^{\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{9}$$

3.5 Expected Value of Research

The leader of sector j, producing the intermediate good at quality q_{t_j}, enjoys monopoly profits from the time he made the discovery τ_{t_j} until the time a competitor will come up with the next innovation q_{t_j+1} at time τ_{t_j+1}. Let Γ_{j,t_j} denote the time interval over which t_j retains the leadership

$$\Gamma_{j,t_j} = \tau_{t_j+1} - \tau_{t_j}.$$

Denote V_{j,t_j} the present value of this stream of profits. Assuming r constant

$$V_{j,t_j} = \int_0^{\Gamma_{j,t_j}} \pi_{j,t_j} e^{-rs} ds.$$

Over the period Γ_{j,t_j} the profits are constant: both q_{t_j} and w_{j,t_j} do not vary. Solving for the benefits from the t_j innovation, we get that they increase with the amount of profits and their duration

$$V_{j,t_j} = \frac{\pi_{j,t_j}(1 - e^{-r\Gamma_{j,t_j}})}{r}. \tag{10}$$

However, the duration Γ_{j,t_j} is a random variable, so what interests the intermediate producer is actually the expected benefit $E(V_{j,t_j})$. In order to determine this value, we need to derive the probability per unit of time that innovation occurs at a point in time τ. Let us define this probability $g(\tau)$. This density function can be found by computing the derivative of the cumulative probability density function $G(\tau)$ for Γ_{j,t_j}. Then, $G(\tau)$ describes the probability that $\Gamma_{j,t_j} \leq \tau$. The probability per unit of time that innovation occurs at τ is equal to the probability of a discovery occurring per unit of time δ_{j,t_j} (Eq. 4) conditional on a discovery not having happened so far $[1 - G(\tau)]$

$$g(\tau) = \frac{dG(\tau)}{d\tau} = [1 - G(\tau)]\,\delta_{j,t_j}. \tag{11}$$

Throughout Γ_{j,t_j}, the number of workers in R&D, n_{j,t_j}, is constant, but we cannot argue the same for all the elements of (Eq. 4). Innovations take place at different times in different sectors, making the R&D workforce to react. If a steady state exists, then the optimal research intensity needs to be constant at the optimal level, and the technological difference between sectors would, on average, be constant as well. In the next section we prove that an equilibrium exists, so we can treat the independent variables in (Eq. 4) as constant (on average) during Γ_{j,t_j}. Hence, by treating δ_{j,t_j} as constant, we solve for the differential equation

$$\dot{G}(\tau) + p_{j,t_j} G(\tau) = \delta_{j,t_j},$$

and we define a solution for the density function

$$G(\tau) = 1 - e^{-\delta_{j,t_j}\tau},$$

such that

$$g(\tau) = \dot{G}(\tau) = \delta_{j,t_j} e^{-\delta_{j,t_j}\tau}.$$

This last result allows us to compute the expected benefits of research

$$\begin{aligned} E(V_{j,t_j}) &= E\left(\frac{\pi_{j,t_j}(1-e^{-r\Gamma_{j,t_j}})}{r}\right) = \int_0^\infty \frac{\pi_{j,t_j}(1-e^{-r\tau})}{r} \delta_{j,t_j} e^{-p_{j,t_j}\tau} d\tau \\ &= \frac{\pi_{j,t_j}}{r+\delta_{j,t_j}}. \end{aligned} \tag{12}$$

Plugging (Eq. 9) into (Eq. 12) gives the full expression for the expected value of successful research. A few comments may be drawn from (Eq. 12). The denominator of this expression displays the obsolescence adjusted interest rate, namely $r + \delta_{j,t_j}$, where the term δ_{j,t_j} represents the effect of creative destruction. When the amount of research conducted by the neighbors and their relative technological advancements are constant, a high level of research in sector j diminishes the expected payoffs from innovation in sector j. Generally, a highly innovative sector displays higher probability of obsolescence too. This causes the monopoly profits to be enjoyed for a shorter period of time. Similarly, assuming constant the level of research in sector j, high contemporaneous level of research from the neighbors increases the probability of successful research in sector j. As before, this process shortens the length of the monopoly profit for the incumbent producer in sector j, and, as a consequence, it decreases the expected value of research, namely the payoff from innovation.

3.6 Determination of R&D Effort

As explained above, the uncertainty related to the R&D outcomes is proportional to the probability that an innovation occurs, which is a random event at the already mentioned arrival rate δ_{j,t_j}. In order to be the monopolist of invention t_j+1, the cost of research has to be born during innovation period t_j, while the benefits, if ever, are going to be enjoyed during invention period t_j+1.

The cost of research per unit of time during invention period t_j is the amount paid to the employees in research. Taking into account the normalization of wages, the cost is equal to n_{j,t_j}. The benefit of an innovation can be calculated as the product between the probability of success per unit of time, δ_{j,t_j}, and the enjoyment of $E(V_{j,t_j})$ during invention period t_j+1 (i.e. $\delta_{j,t_j}E(V_{j,t_j+1})$).

In equilibrium, $n_{j,t_j}\geq 0$, for each potential innovator the cost of an additional unit of research should be equalized to the benefit $\delta_{j,t_j}E(V_{j,t_j+1})$, i.e. the value of becoming the t_j+1 monopolist times the probability of innovation success of an extra unit of research

$$\lambda(\frac{q^{t_{j-1}}}{q^{t_j}},n_{j-1,t_{j-1}},\frac{q^{t_{j+1}}}{q^{t_j}},n_{j+1,t_{j+1}})E(V_{j,t_j+1})\leq 1, \qquad n_{j,t_j}\geq 0. \tag{13}$$

High levels of contemporaneous research in more advanced neighborhood $j-1$ and $j+1$ increase the productivity of research in sector j. This determines an increase in the value of research, which leads to higher level of research in sector j. Replacing (Eq. 4) and (Eq. 12) into (Eq. 13), the equilibrium condition becomes

$$\frac{1}{\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}})}\geq\frac{\pi_{j,t_j+1}}{r+\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}})n_{j,t_j+1}},$$

with equality holding for at least one sector.

Neighbors' current research decreases the cost of current research. In fact, we are implicitly assuming that the research produced by one firm is a perfect substitute for the research of the other firms, so that the positive spillovers are due to the increased research productivity when neighbors are researching intensively. On the contrary, neighbors' future research decreases the benefit of sector j current research. The cause of this negative effect is that an expected increase in productivity of research increases current research obsolescence rate by shortening the period of time for which monopolist profit can be enjoyed, i.e. it decreases the expected benefit of research.

Rearranging the previous condition by replacing the profit function at period invention t_j+1 yields

$$r+\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}+1},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}+1})n_{j,t_j+1}=\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}})\left(\frac{1-\alpha}{\alpha}\right)L\left[A\alpha^2q^{\alpha}_{t_j+1}\right]^{\frac{1}{1-\alpha}},$$

and applying (Eq. 1) gets

$$\frac{1}{q_{t_j}^{\alpha}} = \frac{\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}}) \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} LA^{\frac{1}{1-\alpha}} q^{\alpha} \left[q_{t_j+1}^{\alpha}\right]^{\frac{-\alpha}{1-\alpha}}}{r + \lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}+1}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}+1}) n_{j,t_j+1}}. \tag{14}$$

At each point in time the labor market condition has to hold, so that

$$l_j - n_{j,t_j} = L \left(A\alpha^2\right)^{\frac{1}{1-\alpha}} \left[q_{t_j}^{\alpha}\right]^{\frac{\alpha}{1-\alpha}}$$

which put differently

$$\frac{1}{q_{t_j}^{\alpha}} = L^{(1-\alpha)} A\alpha^2 \left(l_j - n_{j,t_j}\right)^{-(1-\alpha)}. \tag{15}$$

Since the amount of research will ultimately depend on neighbors' research and technological progress, the wage will vary too depending on location.

The remainder of the derivation is reported in the Appendix, while the next section reports the results.

4 Spillover Effects and Growth

At the steady state, recalling that $q_{t_j} = q_{t_{j+1}}$ so that $q_{t_{j+1}} = q^* q_{t_j}$ which implies $q^* = 1$, the values of n_{j,t_j} and q_{t_j} correspond to

$$\overline{n_j} = \frac{(1-\alpha)}{2\alpha - 1} \left[l_j - \left(\frac{r}{\left(\frac{1-\alpha}{\alpha}\right) Q} \right)^{\frac{1}{1-c-d}} \right], \tag{16}$$

$$\overline{q}_j = \left\{ L^{(1-\alpha)} A\alpha^2 \left[l_j - \frac{(1-\alpha)}{2\alpha - 1} \left[l_j - \left(\frac{r}{\left(\frac{1-\alpha}{\alpha}\right) Q_j} \right)^{\frac{1}{1-c-d}} \right] \right]^{-(1-\alpha)} \right\}^{\frac{1}{\alpha}}. \tag{17}$$

In particular, since $Q_j \equiv \frac{q_{t_{j-1}}}{q_{t_j}} \frac{q_{t_{j+1}}}{q_{t_j}}$ in every period, we can draw the following interesting considerations. At time t, for $Q_j > 1$, an increase of Q_j implies that the qualities of the two neighbors are better than j.[9] Given the other parameters' values, an increase in Q_j entails an increase in the number of qualified R&D workers in sector j (if $c + d < 1$). Firms in j realize that they have to invest more in R&D for reducing the gap between them and their neighbors in order to achieve monopoly power. This, according to (Eq. 17), causes

[9] In case of negative variation of Q_j, i.e. the quality q_{t_j} is better than the neigbors' ones, the results are symmetric: since the neighbours try to fill the existing gap, sector t_j looses its leading position.

an improvement in q_{t_j} quality level. The change in q_{t_j} is due to the positive spillover that sector j (at time t) receives from the neighbors $j-1$ and $j+1$, and to its own increase in R&D expenses which make an innovation in j more likely to happen.

Considering now the size of a sector (i.e. the value of parameter l_j), an increase in l_j entails an increase of the number of workers devoted to R&D ($\overline{n_j}$). For low values of α (i.e. for $\alpha \leq 2/3$) this increase in the number of qualified workers $\overline{n_j}$ is lower than the increase in the number of workers l_j and quality $\overline{q}_j$ will increase. For high values of α (for $\alpha > 2/3$), the variation of $\overline{q}_j$ with respect to an increase of l_j is ambiguous.

Conclusion 4 *A high level of local spillovers (a high value of Q_j) entails an increase in the number of workers in the R&D sector as well as in the level of the highest quality available in sector j. Similarly, a big size of the sector j (high level of l_j) leads to a high number of workers involved in R&D and, for low values of α, the level of quality will be better.*

According to this model, spillovers are important forces. At the steady state firms will not embark in new innovation to improve the quality of the goods ($q = 1$), even if they may experience simultaneously different level of quality in different sectors. Such differences yield the rise of spillover effects. Given that, we are then interested in determining to what extent spillovers may drive economic growth. In order to answer this, we need to derive aggregate output and total quantity of intermediate goods as function of local spillovers. By aggregating (Eq. 8) across intermediate sectors, we get

$$X = A^{1/(1-\alpha)}\alpha^{\frac{2}{1-\alpha}}L\sum_{j=1}^{m}(\overline{q}_{t_j})^{\frac{\alpha}{1-\alpha}}$$

Plugging (Eq. 8) into (Eq. 2) and aggregating over firms i, we obtain the expression for aggregate output

$$Y = A^{1/(1-\alpha)}\alpha^{\frac{2\alpha}{1-\alpha}}L\sum_{j=1}^{m}(\overline{q}_{t_j})^{\frac{\alpha}{1-\alpha}}. \tag{18}$$

In equilibrium, after algebraic manipulation of (Eq. 8), we get that $(\overline{q}_{t_j})^{\frac{\alpha}{1-\alpha}} = LA^{1/(1-\alpha)}\alpha^{\frac{2}{1-\alpha}}(\overline{x}_j)$,where $\overline{x}_j = (l_j - \overline{n_j})$. At the steady state, since for every firm i q_{t_j} is equal to (Eq. 17), the quantity depends on the aggregate index of the spillover parameter Q_j.

Local spillovers may be a source of growth, but, since they can be positive or negative according to the size of $\left(q_{t_{j+1}}, q_{t_j}, q_{t_{j+1}}\right)$, we need to pay attention to this in order to derive the conditions for growth. To obtain a finite analytical solution, let's concentrate on the case where $\alpha = 2/3$ and $c = d = 0$. Under these assumptions,[10] according to (Eq. 16) and (Eq. 17)

[10] If $c+d < 1$ instead, we would get $\gamma_x = -\frac{1}{1-c-d}\gamma_Q$, i.e. the relationship between the two rates of growth are magnified by a positive constant.

$$\gamma_Y = \gamma_X = \gamma_{\overline{Q}}.$$

with the aggregate index of spillovers $\overline{Q} = \left\{ \sum_{j=1}^{m} [Q_j]^{-1} \right\}$ where $Q_j \equiv \frac{q_{t_{j-1}}}{q_{t_j}} \frac{q_{t_{j+1}}}{q_{t_j}}$.

For the specified parameters' values, $\overline{n_j}$ reduces so that

$$\overline{x}_j = \frac{2r}{Q_j}$$

To obtain the values of Y and X we aggregate $\overline{x}_j$ and find that they are both function of the aggregate spillover index $\overline{Q}$.[11] In order to quantify how spillovers affect growth, we need to derive the rate of growth of spillovers at the steady state. Again, to deal with tractable solutions, we concentrate on the case in which Q_j is constant across sectors. This assumption implies we can simplify the aggregate spillover index to $\overline{Q} = mQ_j$. By the use of logarithms and derivatives in the expression of $\overline{Q}$, and exploiting (Eq. 1), we obtain that at the steady state

$$\gamma_{\overline{Q}} = \left[\frac{2}{q_{t_j}} - \frac{1}{q_{t_{j+1}}} - \frac{1}{q_{t_{j-1}}} \right]. \tag{19}$$

Expression (Eq. 19) shows that when sector j is a net receiver of spillovers from its two neighbors (namely $\left\{ q_{t_{j+1}}, q_{t_{j-1}} \right\} > q_{t_j}$),[12] its quality improves and $\gamma_{\overline{Q}} > 0$. Assuming that the level of quality in each sector at time t is greater than one, by computing the rate of growth of X via (Eq. 3), we obtain that γ_x is positive too. Hence, according to (Eq. 2), once a sector benefits from positive spillovers, γ_x is expected to be positive as well as γ_Y.

Conclusion 5 *At the steady state, for $\alpha = 2/3$, when each sector enjoys positive spillovers from the neighbors, the rate of growth of the system is positive.*

4.1 Households Spending and Welfare Analysis

So far, we haven't explicitly considered a particular form for the consumers' utility function. In fact, we implicitly assumed that in this closed economy the total quantity of the final good (Y) is completely consumed by the population ($\overline{L}$). As in Barro and Sala-i Martin (1995), we now impose the assumption of a constant intertemporal elasticity function (CIES). Therefore, the problem for the households is to maximize the utility function

[11] We remind the reader that the interest rate r is taken as constant. Moreover, at the steady state, the differences of qualities between sectors are constant, so the aggregate index is the sum of the spillovers effects between each couple of sectors.

[12] This conclusion is involved by the assumption that t_j, t_{j-1}, and t_{j+1} represent the latest quality rung in sector j, $j-1$, and $j-1$ respectively, at the same point in time they are all different. Or, put differently, the same quality steps are reached by the sectors at different points in time.

$$U = \int_0^{\infty} \left(\frac{c^{1-\theta} - 1}{1-\theta} \right) e^{-\rho t} dt, \tag{20}$$

where c represents consumption, ρ is the rate of intertemporal preference and the elasticity of the marginal utility is -θ. Households aggregate income is the sum of the wages (over the fixed aggregate quantity of labor) and the interests on the total assets (market values of all firms). Households optimization problem gives the standard condition for the growth rate of consumption

$$\gamma_c = \frac{1}{\theta}(r - \rho). \tag{21}$$

In order to compute the steady state conditions, we assumed that the interest rate was given as constant. Since in our setting at the steady state $Y = C$, we derive that $\gamma_c = \gamma_Q$. By comparing (Eq. 19) with (Eq. 21), we obtain that

$$r = \theta \left[\frac{2}{\overline{q}_j} - \frac{1}{\overline{q}_{j-1}} - \frac{1}{\overline{q}_{j+1}} \right],$$

which is constant in equilibrium for the given equilibrium values of the parameters ($\overline{q}_j$, $\overline{q}_{j-1}$, $\overline{q}_{j+1}$).

We now turn to assess the level of welfare when local spillovers are present. At the steady state, from the account equation (Eq. 16) it is straightforward to compute

$$\overline{x}_j = \frac{3\alpha - 2}{2\alpha - 1} l_j + \frac{1-\alpha}{2\alpha - 1} \left[\frac{r}{\frac{1-\alpha}{\alpha} Q_j} \right]^{\frac{1}{1-c-d}}. \tag{22}$$

From the previous equation, via $(\overline{q}_{t_j})^{\frac{\alpha}{1-\alpha}} = LA^{1/(1-\alpha)} \alpha^{\frac{2}{1-\alpha}} (\overline{x}_j)$, we can compute the value of q_{t_j}, which plugged into (Eq. 18), and after aggregating across sectors, yields

$$Y = A^{2/(1-\alpha)} \alpha^{\frac{2(\alpha+1)}{1-\alpha}} L^2 \left\{ \sum_{j=1}^{m} \left[\left(\frac{3\alpha - 2}{2\alpha - 1} \right) l_j + \frac{1-\alpha}{2\alpha - 1} \left(\frac{\alpha r}{(1-\alpha) Q_j} \right)^{\frac{1}{1-c-d}} \right] \right\}. \tag{23}$$

From (Eq. 23) we derive that positive variations of Q_j cause Y to decrease, and viceversa. Since Y can also be interpreted as the flow of consumption goods, its values will be high for low levels of Q_j. Stated differently, the level of welfare increases when the disparities across sectors are small, namely when the size of the spillovers is not too high.

Conclusion 6 *Consumers are better off with low levels of spillovers .*

This last conclusion points to a clear trade-off regarding the nature and the effects of local spillovers. On one hand, positive spillovers improve the quality of local goods. On the other, a high level of positive spillovers causes high discrepancies across intermediate sectors which induce a decrease in the level of households welfare (intended as the steady state flow of consumption). Therefore, policies that stimulate and support collaborations among firms (so to reduce the quality gap) are desirable to improve the welfare of the system, even if reducing such a gap causes a smoothing of the magnitude of the equilibrium growth rate.

5 Conclusion

This study addresses the role of local technological spillovers in driving economic growth. In this model we assume that sectors (and firms) are located around a circle and positive spillovers may occur between neighbors. In particular, these positive spillovers concern the R&D activity and the likelihood to obtain an innovation when locating next to other firms investing in R&D.

One important conclusion is that spillovers may have ambiguous effects. Keeping constant the research level of a particular sector, a high level of research in its neighborhood increases the probability of successful innovation, but also the obsolescence rate. In addition, neighbors' future research decreases the benefits of that sector's current research. In addition, we found that the spillovers' growth rate drives the growth rate of the economy (for specified parameters' values). Under the stated assumptions, we can assert that when the economy as a whole (summing up supplier and receiver sectors) is a net receiver of positive spillovers, spillovers foster economic growth. The opposite happens when sectors, as a whole, are net suppliers of spillovers.

Bearing these conclusions in mind, the policy implications of this model are that policymakers could profitably sustain the creation of agglomerations (or networks) to foster local growth—provided the prevention of leading positions among firms belonging to the agglomeration. Hence, policies that sustain sharing of knowledge across group members should be useful in guaranteeing an active role for local agglomerations. However, recalling the results of welfare analysis, we should add that high discrepancies among the quality of goods supplied by firms entail also a negative effect on local welfare, since high levels of spillovers yield proportionally low level of consumptions.

Further development of this analysis could focus on the welfare effects in order to understand the extent to which local agglomeration helps local development. Additionally, an interesting extension would be to study the formation of local agglomerations in an economic setting with local spillovers in which the location of the firms is not exogenously given.

References

[1] Aghion, P. and Howitt, P., 1992, "A Model of Growth through Creative Destruction", *Econometrica*, vol. 60(2), pp.323-351.

[2] Audretsch, D.B. and Feldman, M. (1996): "R&D Spillovers and the Geography of Innovation and Production", *American Economic Review,* vol.86839, pp.630-640.

[3] Barro, R. and Sala-I-Martin, X., 1995, *Economic Growth,* McGraw-Hill.

[4] Basevi, G. and Ottaviano G.I.P., 2002, "The district and the global economy: exportation versus foreign location*", Journal of Regional Science,* vol. 42(1), pp. 107-126.

[5] Cooke, Ph. and Morgan, K., 1998: '*The Associational Economy*', Oxford University Press.

[6] Feldman, M and Audretsch, D.B., 1999: "Innovation in cities: Science-based diversity, specialization and localized competition", *European Economic Review,* vol.43, pp. 409-429.

[7] Fujita M., Krugman P., and Venables, 1999: "*The Spatial Economy : Cities, Regions, and International Trade"*, MIT Press.

[8] Fujita M. and Thisse, J.F., 2002, "*Economics of Agglomerations",* Cambridge University Press.

[9] Grossman, G. and Helpman, E, 1991a, "Quality Ladders and Product Cycles", *The Quarterly Journal of Economics*, vol.106(2), pp. 557-586.

[10] Grossman, G. and Helpman, E.,1991b, "*Innovation and Growth in the Global Economy"*, Cambridge, Massachusetts: MIT Press.

[11] Lucas, R.,1988, "On the mechanism of economic development", *Journal of Monetary Economics*, vol.22, pp. 3-22.

[12] Krugman, P.,1991: "*Geography and Trade"*, MIT Press.

[13] Nicolini, R., 2002: "*R&D et développement régional en Belgique: quelques perspectives*', in Services Fédéraux des affaires scientifiques, techniques et culturelles. Rapport belge en matière de science, technologie et innovation, 2001, pag. 145-171, Bruxelles, Belgique,

[14] OECD,1996: '*Network of Enterprises and Local Development - Competing and Co-operating in Local Productive Systems'*, Local Economic and Employment Development Series

[15] Ottaviano, G. I. P. and Thisse, J.F., 2003: "*Agglomeration and economic geography*", CORE Discussion Paper n. 2003-16.

[16] Peri, G., 2003: "*Knowledge Flows, R&D Spillovers and Innovation",* ZEW Discussion Paper, n.40/03.

[17] ______, 2004: "*Determinants of Knowledge Flows and their Effects on Innovation*", mimeo, University of Davis.

[18] Romer, P.M., 1990: "Endogenous Technological Change", *Journal of Political Economy*, vol.98(5), part II, pp. S71-S102.

[19] Rosenthal, S.S. and Strange, W.C. (2003): "Geography, industrial organization, and agglomeration", *The Review of Economics and Statistics*, vol. 85(2), pp. 377-393.

[20] RUR-CENSIS, 2001, " Rapporto FEDERCOMIN. I Distretti Produttivi Digitali".

[21] Saxenian, A.,1994: "*Regional Advantage: Culture and Competition in Silicon Valley and Route 128",* Harvard University Press.

[22] Salop, S., 1979: "Monopolistic competition with outside goods", *Econometrica*, vol.10(1), pp. 141-156.

A Appendix

First, it is important to remember that t_j is not a particular instant in time, but the latest quality step in sector j, so that, for different sectors, t_j, t_{j-1}, t_{j+1} will signify different numbers. Combining (Eq. 14) and (Eq. 15), we get

$$L^{(1-\alpha)}A\alpha^2\left(l_j-n_{j,t}\right)^{-(1-\alpha)}=$$

$$=\frac{\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}})\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}}LA^{\frac{1}{1-\alpha}}q^{\alpha}\left[L^{(1-\alpha)}A\alpha^2\left(l_j-n_{j,t_j+1}\right)^{-(1-\alpha)}\right]^{\frac{-\alpha}{1-\alpha}}}{r+\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}+1},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}+1})n_{j,t_j+1}},$$

that can be reduced to

$$\frac{1}{\left(l_j-n_{j,t_j}\right)^{(1-\alpha)}}=\frac{\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}})\left(\frac{1-\alpha}{\alpha}\right)q^{\alpha}\left(l_j-n_{j,t_j+1}\right)^{\alpha}}{r+\lambda(\frac{q^{t_{j-1}+1}}{q^{t_j+1}},n_{j-1,t_{j-1}+1},\frac{q^{t_{j+1}+1}}{q^{t_j+1}},n_{j+1,t_{j+1}+1})n_{j,t_j+1}}. \quad (24)$$

An anticipated increase in the research efforts in one sector discourages current research (*creative destruction effect*), shortening the expected lifetime of innovation monopoly.

Let us define $\phi \equiv \left(\frac{1-\alpha}{\alpha}\right) q^{\alpha}$, (Eq. 24) becomes

$$r + \lambda(\frac{q^{t_{j-1}+1}}{q^{t_j+1}}, n_{j-1,t_{j-1}+1}, \frac{q^{t_{j+1}+1}}{q^{t_j+1}}, n_{j+1,t_{j+1}+1}) n_{j,t_j+1} =$$

$$= \phi\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}}) \left(l_j - n_{j,t+1}\right)^{\alpha} \left(l_j - n_{j,t}\right)^{(1-\alpha)}.$$

Since t_j, t_{j-1}, and t_{j+1} represent the latest quality rung in sector j, $j-1$, and $j+1$, at the same point in time they are all different, i.e. the same quality steps are reached by different sectors at different point in time.

Let's now assume a specific form for (Eq. 4) such that

$$\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}}) \equiv \left(\frac{q_{t_{j-1}}}{q_{t_j}} \frac{q_{t_{j+1}}}{q_{t_j}}\right) \vartheta(n_{j-1,t_{j-1}}, n_{j+1,t_{j+1}}),$$

and

$$\lambda(\frac{q^{t_{j-1}+1}}{q^{t_j+1}}, n_{j-1,t_{j-1}+1}, \frac{q^{t_{j+1}+1}}{q^{t_j+1}}, n_{j+1,t_{j+1}+1}) \equiv \left(\frac{q^{t_{j-1}+1}}{q^{t_j+1}} \frac{q^{t_{j+1}+1}}{q^{t_j+1}}\right) \vartheta(n_{j-1,t_{j-1}+1}, n_{j+1,t_{j+1}+1}).$$

Since for sector j is t is an important determinant, we define $\vartheta(\tau) \equiv \vartheta(n_{j-1,t_{j-1}}, n_{j+1,t_{j+1}})$ and $\vartheta(\tau+1) \equiv \vartheta(n_{j-1,t_{j-1}+1}, n_{j+1,t_{j+1}+1})$ in order to determine to what extent the amount of contemporaneous neighbors' research affect sector j. Hence, the equilibrium condition becomes

$$\left(l_j - n_{j,t_j}\right) = \left[\frac{r + \left(\frac{q^{t_{j-1}+1}}{q^{t_j+1}} \frac{q^{t_{j+1}+1}}{q^{t_j+1}}\right) \vartheta(\tau+1) n_{j,t_j+1}}{\phi \left(\frac{q_{t_{j-1}}}{q_{t_j}} \frac{q_{t_{j+1}}}{q_{t_j}}\right) \vartheta(\tau) \left(l_j - n_{j,t_j+1}\right)^{\alpha}}\right]^{\frac{1}{(1-\alpha)}},$$

which after substituting $x_t = \left(l_j - n_{j,t_j}\right)$ yields

$$x_t = \left[\frac{r + \left(\frac{q^{t_{j-1}+1}}{q^{t_j+1}} \frac{q^{t_{j+1}+1}}{q^{t_j+1}}\right) \vartheta(\tau+1)(l_j - x_{t+1})}{\phi \left(\frac{q_{t_{j-1}}}{q_{t_j}} \frac{q_{t_{j+1}}}{q_{t_j}}\right) \vartheta(\tau) x_{t+1}^{\alpha}}\right]^{\frac{1}{(1-\alpha)}}.$$

It can be easily checked that when neighbors' research intensity does not vary, $\vartheta(\tau) = \vartheta(\tau+1)$, sector j research intensity present a unique crossing with the steady state line in the set of existence of the function.

We can conclude that a steady state exists for which $x_t = x_{t+1}$. This implies that for all sectors j, $j = 0, 1, 2...m-1$, it holds that $n_{j,t_j} = n_{j,t_j+1} = \overline{n_j}$, which implies

$\vartheta(\tau) = \vartheta(\tau+1) = \vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})$, while q_{t_j}, $q_{t_{j-1}}$ ... at each point in time are considered exogenously given in each sector.

The steady state equation becomes

$$r + \left(\frac{q_{t_{j-1}+1}}{q_{t_j+1}}\frac{q_{t_{j+1}+1}}{q_{t_j+1}}\right)\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})\overline{n_j} = \phi\left(\frac{q_{t_{j-1}}}{q_{t_j}}\frac{q_{t_{j+1}}}{q_{t_j}}\right)\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})\left(l_j - \overline{n_j}\right).$$

We define $Q \equiv \frac{q_{t_{j-1}}}{q_{t_j}}\frac{q_{t_{j+1}}}{q_{t_j}}$, and via (Eq. 1) we deduce even that $Q = \frac{q_{t_{j-1}+1}}{q_{t_j+1}}\frac{q_{t_{j+1}+1}}{q_{t_j+1}}$, and we get

$$\overline{n_j} = \frac{\phi l_j \vartheta(\overline{n}_{j-1}, \overline{n}_{j+1}) - rQ^{-1}}{(1+\phi)\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})}.$$

In order to find a finite solution for $\overline{n_j}$, $\overline{n}_{j-1}$, and $\overline{n}_{j+1}$ we need to specify the function $\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})$ and solve for the difference equation

$$\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})\left[l_j - \frac{(1-\phi)}{\phi}\overline{n_j}\right] = \frac{r}{\phi}Q^{-1}$$

The solution method requires a linearization of the previous expression, so we assume the following convenient functional form for $\vartheta(\cdot)$

$$\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1}) = \left[l_{j-1} - \frac{(1-\phi)}{\phi}\overline{n}_{j-1}\right]^{-d}\left[l_{j+1} - \frac{(1-\phi)}{\phi}\overline{n}_{j+1}\right]^{-c},$$

with $c, d \geq 0$ in order to satisfy the assumption that an increase in neighbors research intensity generate a positive technological spillover. Taking logs and defining $z_j \equiv \log\left[l_j - \frac{(1-\phi)}{\phi}\overline{n_j}\right]$ and $\xi \equiv \log\left(\frac{r}{\phi}Q^{-1}\right)$, the second order difference equation becomes:

$$-cz_{j+2} + z_{j+1} - dz_j = \xi.$$

The particular integral is

$$z_p = \frac{\xi}{1-c-d},$$

and since ξ is positive, in order to z_p to be positive we need to impose $c + d < 1$.

The characteristic equation presents the roots: $b_{1,2} = \frac{1}{2c} \pm \frac{1}{2c}\sqrt{1-4cd}$. In order to have two distinct real roots, c and d have to satisfy $1 - 4cd \geq 0$ or $cd \leq \frac{1}{4}$. Since the previous restriction was $c + d < 1$, this second condition will be always satisfied, and the general solution becomes

$$z_j = \frac{\xi}{1-c-d} + A_1\left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^j + A_2\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^j.$$

Imposing the initial condition $z_0 = \frac{\xi}{1-c-d} + A_1 \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^0 + A_2 \left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^0$ we find

$$A_1 = z_0 - \frac{\xi}{1-c-d} - A_2.$$

Imposing periodicity (i.e. if we have m sectors arrayed along a circle it must be that the first one coincides with the one after the last one, so if z_0 is the first one z_{m-1} is the last one, and the condition becomes $z_0 = z_m$)

$$\frac{\xi}{1-c-d} + A_1 + A_2 = \frac{\xi}{1-c-d} + A_1 \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m + A_2 \left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m,$$

and from this, after substituting the previously found A_1, we obtain

$$A_1 = \left(z_0 - \frac{\xi}{1-c-d}\right) \left[\frac{-1 + \left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m}{\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m - \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m}\right],$$

and

$$A_2 = \left(z_0 - \frac{\xi}{1-c-d}\right) \left[\frac{1 - \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m}{\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m - \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m}\right],$$

so the solution becomes

$$\begin{aligned} z_j \quad = \quad & \frac{\xi}{1-c-d} + \left(z_0 - \frac{\xi}{1-c-d}\right) \left[\frac{-1 + \left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m}{\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m - \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m}\right] \cdot \\ & \cdot \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^j + \left(z_0 - \frac{\xi}{1-c-d}\right) \cdot \\ & \cdot \left[\frac{1 - \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m}{\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m - \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m}\right] \left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^j. \end{aligned}$$

Solving for z_m, we obtain $z_m = \frac{\xi}{1-c-d}$, so it must be true that $z_0 = \frac{\xi}{1-c-d}$, which implies for our solution that

$$z_j = \frac{\xi}{1-c-d} \qquad \forall j = 0, 1, 2, ..., m-1.$$

Recalling $z_j \equiv \log\left[l_j - \frac{(1-\phi)}{\phi}\overline{n_j}\right]$ and $\xi \equiv \log\left(\frac{r}{\phi}Q^{-1}\right)$, the solution for the optimal value of research intensity is

$$\overline{n_j} = \frac{\phi}{(1-\phi)} \left[l_j - \left(\frac{r}{\phi}Q^{-1}\right)^{\frac{1}{1-c-d}}\right],$$

and since $\phi \equiv \left(\frac{1-\alpha}{\alpha}\right) q^\alpha$, the solution becomes:

$$\overline{n_j} = \frac{(1-\alpha)q^{\alpha}}{\alpha-(1-\alpha)q^{\alpha}}\left[l_j - \left(\frac{r}{\left(\frac{1-\alpha}{\alpha}\right)q^{\alpha}Q}\right)^{\frac{1}{1-c-d}}\right],$$

$$\overline{q}_{t_j} = \left\{L^{(1-\alpha)}A\alpha^2\left[l_j - \frac{(1-\alpha)q^{\alpha}}{\alpha-(1-\alpha)q^{\alpha}}\left[l_j - \left(\frac{r}{\left(\frac{1-\alpha}{\alpha}\right)q^{\alpha}Q}\right)^{\frac{1}{1-c-d}}\right]\right]^{-(1-\alpha)}\right\}^{\frac{1}{\alpha}}.$$